

Collision Avoidance for Bi-Steerable Car Using Analytic Left Inversion

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Abstract—A case study is presented of a collision avoidance system that directly integrates the kinematics of a bi-steerable car with a suitable path planning algorithm. The first step is to identify a path using the method of rapidly exploring random trees, and then a spline approximation is computed. The second step is to solve the output tracking problem by explicitly computing the left inverse of the kinematics of the system to render the Taylor series of the desired input for each polynomial section of the spline approximation. The method is demonstrated by numerical simulation.

Index Terms—Collision avoidance, motion planning, autonomous vehicles, system inversion, Chen-Fliess series

I. INTRODUCTION

A convergence of technological advances in conjunction with changing societal attitudes suggests that autonomous wheeled vehicles are poised to soon enter main stream applications in business, the government, and the private realm [10]. As this happens, it is likely that vehicle design will evolved in new directions once the human driver and passengers are completely replaced by a computer control system and cargo. For example, a bi-steerable car with independent front and rear axels provides improved maneuverability and handling, which would be invaluable in urban settings to improve collision avoidance [11], [12], [19]–[22], [25]. But such improved performance is predicated on control methodologies that properly combine the true dynamics of the vehicle with the proper path planning tools [1], [16].

In this paper, a case study is presented of a collision avoidance system that directly integrates the kinematics of a bi-steerable car with a suitable path planning algorithm. Here collision avoidance means simply steering a vehicle modeled as a point from a starting location to a final location while avoiding all stationary obstacles in between the two locations. The first step is solving the path planning problem. Numerous algorithms appear in the literature which are applicable to the collision avoidance problem, such as rapidly exploring random trees (RRT), probabilistic roadmap (PRM), artificial potential fields, and genetic algorithms [16]. The second step is solving the output tracking problem. That is, determine the control inputs so that the vehicle accurately follows the desired path. Mathematically, this corresponds to computing a left inverse for the system. There is an extensive literature on this topic in the context of geometric nonlinear control theory, see, for example, [3], [5], [13]–[15], [23], [24]. This class of techniques generates left inverses dynamically by driving a certain inverse dynamical system with the desired output. While very convenient in some situations, this method has certain limitations, like requiring the system to be minimum phase. In addition, this approach

does not provide an *explicit* representation of the control input one is seeking. Another common approach is to use the property of *flatness*, which allows one to explicitly compute a left inverse analytically provided suitable flat outputs can be identified [4], [17]. Unfortunately, these output are sometimes not the most desirable from a physical point of view. This turns out to be the case for the bi-steerable car, where the flat outputs do not correspond to simple physical variables like a center point on the vehicle or its speed or orientation [12], [20]–[22]. This then makes integrating the tracking problem with path planning problem more complex. A final inversion method, which is the one employed in this paper, is to use an analytic expression for the left inverse which is available for systems which have a well defined vector relative degree for the outputs of interest [7], [8]. They do *not* have to be flat outputs. The method is based on a Fliess operator representation of input-output system, and a combinatorial Hopf algebra technique that renders an explicit formula for the Taylor series of the left inverse when the output is an analytic function in the range of the input-output map. This single formula can be pre-computed efficiently off-line to arbitrary precision [2], [6], and, for example, could be hardwired on an FPGA for real-time implementation. In this setting, the desired output trajectory is approximated by a spline, and then the left inverse of each polynomial section is numerically evaluated using the inversion formula. It will be shown here that this strategy provides a feasible and accurate solution to the collision avoidance problem for a bi-steerable car. For brevity, only the vehicle's kinematics are considered.

The paper is organized as follows. In Section II, preliminaries concerning Fliess operators and their inverses are briefly summarized to make the presentation more self-contained. Then, in Section III, the left inverse of the kinematics of the bi-steerable car is computed. This result is then integrated in Section IV with the RRT path planning algorithm and the corresponding numerical simulations are presented. Section V provides the paper's conclusions.

II. PRELIMINARIES

In this section, some preliminaries concerning the theory of Fliess operators are outlined as they provide the cornerstone for the method presented in this paper. The interested reader is referred to [8], [9] for a more complete treatment.

A. Fliess Operators and Their Interconnections

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X , $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X . The *length* of η , $|\eta|$, is the number of letters in η . The set of all words with length k is denoted by X^k . The set of all words including the empty word, \emptyset , is designated by X^* . It forms a monoid under catenation. The set ηX^* is comprised of all words

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with the prefix η . Any mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c . Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta) \eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c , $\text{supp}(c)$, is the set of all words having nonzero coefficients. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. It forms an associative \mathbb{R} -algebra under the catenation product and a commutative and associative \mathbb{R} -algebra under the shuffle product, denoted here by \sqcup . The latter is the \mathbb{R} -bilinear extension of the shuffle product of two words, which is defined inductively by $(x_i \eta) \sqcup (x_j \xi) = x_i(\eta \sqcup (x_j \xi)) + x_j((x_i \eta) \sqcup \xi)$ with $\eta \sqcup \emptyset = \emptyset \sqcup \eta = \eta$ for all $\eta, \xi \in X^*$ and $x_i, x_j \in X$.

One can formally associate with any series $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ a causal m -input, ℓ -output operator, F_c , in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$ -norm for a measurable real-valued function, u_i , defined on $[t_0, t_1]$. Let $L_{\mathfrak{p}}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \rightarrow C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0) d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output system corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0).$$

If there exist real numbers $K_c, M_c > 0$ such that $|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!$, $\forall \eta \in X^*$, then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) The set of all such *locally convergent* series is denoted by $\mathbb{R}_{LC}^\ell \langle\langle X \rangle\rangle$. On the other hand, if $|(c, \eta)| \leq K_c M_c^{|\eta|}$, $\forall \eta \in X^*$, then the operator is well defined over $[0, T]$ for all $R, T > 0$. These are called *globally convergent* series, and the set of all such series is denoted by $\mathbb{R}_{GC}^\ell \langle\langle X \rangle\rangle$. A Fliess operator F_c defined on $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ is said to be *realized* by a state space realization

$$\dot{z} = g_0(z) + \sum_{i=1}^m g_i(z) u_i, \quad z(t_0) = z_0 \quad (1a)$$

$$y = h(z), \quad (1b)$$

where each g_i is an analytic vector field expressed in local coordinates on some neighborhood $\mathcal{W} \subseteq \mathbb{R}^n$ of z_0 , and h is an analytic function on \mathcal{W} , if (1a) has a well defined solution $z(t)$, $t \in [t_0, t_0 + T]$ on \mathcal{W} for any given input $u \in B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$, and

$$F_c[u](t) = h(z(t)), \quad t \in [t_0, t_0 + T].$$

In this case, the coefficients of the i -th component of generating series c are computed by

$$(c_i, \eta) = L_{g_\eta} h_i(z_0), \quad \eta \in X^*, \quad (2)$$

where

$$L_{g_\eta} h_i := L_{g_{j_1}} \cdots L_{g_{j_k}} h_i, \quad \eta = x_{j_k} \cdots x_{j_1},$$

the *Lie derivative* of h_i with respect to g_j is defined as

$$L_{g_j} h_i : W \rightarrow \mathbb{R} : z \mapsto \frac{\partial h_i}{\partial z}(z) g_j(z),$$

and $L_{g_0} h_i = h_i$.

When Fliess operators F_c and F_d are connected in a parallel-product fashion, it is known that $F_c F_d = F_{c \sqcup d}$. If F_c and F_d with $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ are interconnected in a cascade manner, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where $c \circ d$ denotes the *composition product* of c and d as described in [8]. This product is associative and \mathbb{R} -linear in its left argument c . In the event that two Fliess operators are interconnected to form a feedback system, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d , denoted by $c \circledast d$. This product can be explicitly computed via Hopf algebra methods. The basic idea is to consider the set of operators $\mathcal{F}_\delta = \{I + F_c : c \in \mathbb{R}^m \langle\langle X \rangle\rangle\}$, where I denotes the identity map, as a group under composition. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for \mathcal{F}_δ is denoted by $\mathbb{R} \langle\langle X_\delta \rangle\rangle$. This set also forms a group under the composition product induced by operator composition, namely, $c_\delta \circ d_\delta := \delta + d + c \circ d$, where \circ denotes the *modified composition product* [8]. The group $(\mathbb{R} \langle\langle X_\delta \rangle\rangle, \circ, \delta)$ has coordinate functions that form a Faà di Bruno type Hopf algebra. In which case, the group (composition) inverse $c_\delta^{\circ-1}$ can be computed efficiently via the antipode of this Hopf algebra [2], [6], [8]. This inverse also provide an explicit expression for the feedback product, namely, $c \circledast d = c \circ (\delta - d \circ c)^{\circ-1}$.

B. Left Inversion of Multivariable Fliess Operators

It was shown in [26] that F_c will map every input which is analytic at t_0 to an output which is also analytic at t_0 provided $c \in \mathbb{R}_{LC}^\ell \langle\langle X \rangle\rangle$. In [7] an explicit formula was developed for calculating the left inverse of a multivariable mapping F_c given a real analytic function in its range. Without loss of generality assume $t_0 = 0$. Note that every $c \in \mathbb{R} \langle\langle X \rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$. A condition under which the left inverse of F_c exists is provided by the following definition

Definition 1: Given $c \in \mathbb{R}^m \langle\langle X \rangle\rangle$, let $r_i \geq 1$ be the largest integer such that $\text{supp}(c_{F,i}) \subseteq x_0^{r_i-1} X^*$, where $i = 1, 2, \dots, m$. Then the component series c_i has **relative degree** r_i if the linear word $x_0^{r_i-1} x_j \in \text{supp}(c_i)$ for some $j \in \{1, \dots, m\}$, otherwise it is not well defined. In addition, c has **vector relative degree** $r = [r_1 \ r_2 \ \cdots \ r_m]$ if each c_i

has relative degree r_i and the $m \times m$ matrix

$$A = \begin{bmatrix} (c_1, x_0^{r_1-1} x_1) & (c_1, x_0^{r_1-1} x_2) & \cdots & (c_1, x_0^{r_1-1} x_m) \\ (c_2, x_0^{r_2-1} x_1) & (c_2, x_0^{r_2-1} x_2) & \cdots & (c_2, x_0^{r_2-1} x_m) \\ \vdots & \vdots & \ddots & \vdots \\ (c_m, x_0^{r_m-1} x_1) & (c_m, x_0^{r_m-1} x_2) & \cdots & (c_m, x_0^{r_m-1} x_m) \end{bmatrix}$$

has full rank. Otherwise, c does not have vector relative degree.

This notion of vector relative degree agrees with the usual definition in the state space setting [15]. In particular, c has vector relative degree r only if for each i the series $(x_0^{r_i-1} x_j)^{-1}(c_i)$ is non proper for some j . Here the left-shift operator for any $x_i \in X$ is defined on X^* by $x_i^{-1}(x_i \eta) = \eta$ with $\eta \in X^*$ and zero otherwise. Higher order shifts are defined inductively via $(x_i \xi)^{-1}(\cdot) = \xi^{-1} x_i^{-1}(\cdot)$, where $\xi \in X^*$. The left-shift operator is assumed to act linearly and componentwise on $\mathbb{R}^m \langle \langle X \rangle \rangle$. The shuffle inverse of any series $C \in \mathbb{R}^{m \times m} \langle \langle X \rangle \rangle$ is given by

$$C^{\sqcup -1} = ((C, \emptyset)(I - C'))^{\sqcup -1} = (C')^{\sqcup *} (C, \emptyset)^{-1},$$

where $C' = I - (C, \emptyset)^{-1} C$ is proper, i.e., $(C', \emptyset) = 0$, and $(C')^{\sqcup *} := \sum_{k \geq 0} (C')^{\sqcup k}$. The relationship between $C^{\sqcup -1}$ and the multiplicative inverse operator $(F_C)^{-1}$, that is, $F_C(F_C)^{-1} = (F_C)^{-1} F_C = I$, is $(F_C)^{-1} = F_{C^{\sqcup -1}}$.

Let $X_0 := \{x_0\}$, and $\mathbb{R}[[X_0]]$ denotes the set of all commutative series over X_0 . When $c \in \mathbb{R}[[X_0]]$, $F_c[u](t)$ reduces to the Taylor series $\sum_{k \geq 0} (c, x_0^k) E_{x_0^k}[u](t) = \sum_{k \geq 0} (c, x_0^k) t^k / k!$. The main inversion tool used in the paper is given next.

Theorem 1: Suppose $c \in \mathbb{R}^m \langle \langle X \rangle \rangle$ has vector relative degree r . Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}_L^m[[X_0]]$ satisfying $(c_{y_i}, x_0^k) = (c_i, x_0^k)$, $k = 0, 1, \dots, r_i - 1$, $i = 1, 2, \dots, m$. Then the input

$$u(t) = \sum_{k=0}^{\infty} (c_u, x_0^k) \frac{t^k}{k!}, \quad (3)$$

is the unique real analytic solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$, where

$$c_u = \left([C^{\sqcup -1} \sqcup (x_0^r)^{-1}(c - c_y)]^{\circ -1} \right)_N, \quad (4)$$

the i -th row of $(x_0^r)^{-1}(c - c_y)$ is $(x_0^{r_i})^{-1}(c_i - c_{y_i})$, and the (i, j) -th entry of C is $(x_0^{r_i-1} x_j)^{-1}(c_i)$.

Observe that this theorem describes the outputs that can be successfully tracked by F_c , namely, those whose Taylor series coefficients satisfy certain matching conditions. (The same conditions appear in the classical papers on dynamic inversion.) For example, if $m = 2$ and the vector relative degree is $(2, 2)$, the Taylor series of the output are subject to the constraints $(c_{y_1}, \emptyset) = (c_1, \emptyset)$, $(c_{y_2}, \emptyset) = (c_2, \emptyset)$, $(c_{y_1}, x_0) = (c_1, x_0)$, and $(c_{y_2}, x_0) = (c_2, x_0)$, which in turn puts constraints on the class of admissible output paths.

III. LEFT INVERSE OF BI-STEERABLE CAR KINEMATICS

Consider the bi-steerable car shown in Figure 1. For simplicity, only the kinematics are considered. So the car is assumed to have zero mass and move in the plane with the speed of the car $u_1 = \sqrt{\dot{x}^2 + \dot{y}^2}$ and the front axle steering angular velocity $u_2 = \alpha$ as inputs. The kinematics of the

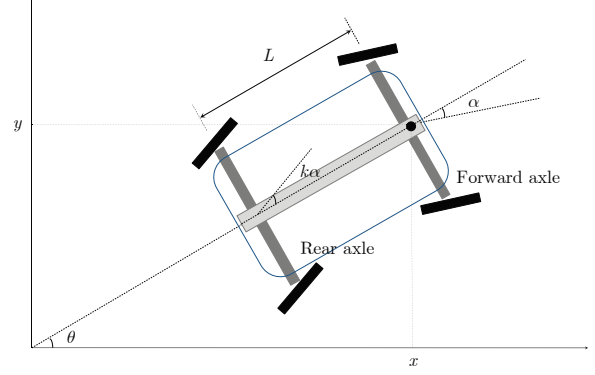


Fig. 1. Bi-steerable car

system are therefore

$$\begin{aligned} \dot{x} &= u_1 \cos(\theta + \alpha), \\ \dot{y} &= u_1 \sin(\theta + \alpha), \\ \dot{\theta} &= u_1 \frac{\sin(\alpha - f(\alpha))}{L \cos(f(\alpha))}, \\ \dot{\alpha} &= u_2, \end{aligned}$$

where $f(\alpha) := k\alpha$ with $k \in \mathbb{R}$. These dynamics assume the usual constraints of rolling without slippage of the wheels, namely,

$$\begin{aligned} 0 &= \dot{y}_{forward} \cos(\theta + \alpha) - \dot{x}_{forward} \sin(\theta + \alpha), \\ 0 &= \dot{y}_{rear} \cos(\theta + f(\alpha)) - \dot{x}_{rear} \sin(\theta + f(\alpha)). \end{aligned}$$

Setting $z_1 = x$, $z_2 = y$, $z_3 = \theta$, $z_4 = \alpha$, the outputs are picked to be the coordinates of the front axle center $y_i = z_i$, $i = 1, 2$. The corresponding two-input, two-output state space realization is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} \cos(z_3 + z_4) \\ \sin(z_3 + z_4) \\ \frac{\sin((1-k)\alpha)}{L \cos(k\alpha)} \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2 \quad (5a)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (5b)$$

Hereafter, the focus is on the $k = -0.7$ case, which as explained in [20] is sufficient for the existence of flat outputs. But this fact is inconsequential here since flat outputs will not be used. The generating series, c , can be computed directly from (2) using the vector fields and output function given in (5):

$$\begin{aligned} c_1 &= z_{1,0} + \cos z_1 - \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \sin x_1^2 - \sin x_1 x_2 \\ &\quad - \cos x_1 x_2 x_2 - \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \cos x_1 x_2 x_1 + \cdots \\ c_2 &= z_{2,0} + \sin x_1 + \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \cos x_1^2 + \cos x_1 x_2 \\ &\quad - \sin x_1 x_2 x_2 - \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \sin x_1 x_2 x_1 + \cdots, \end{aligned}$$

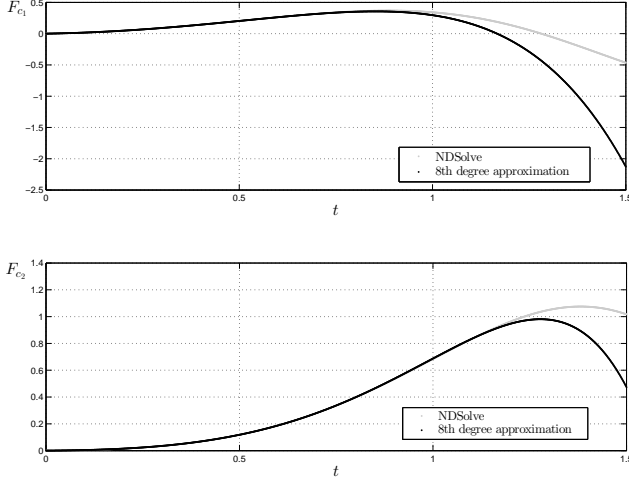


Fig. 2. Mathematica computed (NDSolve) and Fliess operator approximated (degree 8) step response of bi-steerable car ($u_1(t) = 1.5$ and $u_2(t) = 0.8$, $t > 0$)

where $\text{co} := \cos(z_{3,0} + z_{4,0})$ and $\text{si} := \sin(z_{3,0} + z_{4,0})$. In this case, $c \in \mathbb{R}_{GC}^2 \langle \langle X \rangle \rangle$ with $X = \{x_0, x_1, x_2\}$ since all sines and cosines in the numerator of every coefficient can be bounded by 1. Therefore, after some additional calculations a conservative choice for the growth constants is $K_c = \max\{z_{1,0}, z_{2,0}\}$ and $M_c = 2.4 \sec(0.7z_{4,0})$. With $z_{1,0} = z_{2,0} = z_{3,0} = 0$, $z_{4,0} = z_{5,0} = 0.2$, the step response ($u_1(t) = 1.5$ and $u_2(t) = 0.8$, $t > 0$) of the system was computed using the NDSolve Mathematica function [18] and is compared against a degree eight Fliess operator approximation in Figure 2. Since $c \in \mathbb{R}_{GC}^2 \langle \langle X \rangle \rangle$, the output $y = F_c[u]$ is well defined on $[0, \infty)$ for all $u \in B_1^m[0, T](R)$ and any finite $R, T > 0$. [27].

A simple calculation shows that the relative degrees of y_1 and y_2 are both one, but the system does not have a well defined vector relative degree. Applying dynamic extension on u_1 yields the augmented system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{pmatrix} = \begin{pmatrix} z_5 \cos(z_3 + z_4) \\ z_5 \sin(z_3 + z_4) \\ z_5 \frac{\sin((1-k)\alpha)}{L \cos(k\alpha)} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \bar{u}_1, \quad (6)$$

where the new input \bar{u}_1 is taken as the derivative of the original input u_1 . This system has vector relative degree $r = [r_1 \ r_2] = [2 \ 2]$ with $r_1 + r_2 = 4 < 5 = n$ and generating series

$$\begin{aligned} c_1 &= z_{1,0} + z_{5,0} \text{co} x_0 + z_{5,0} \text{si} x_0 x_1 + \text{co} x_0 x_2 \\ &\quad - z_{5,0}^2 \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \text{si} x_0^2 + \dots \\ c_2 &= z_{2,0} + z_{5,0} \text{si} x_0 + z_{5,0} \text{co} x_0 x_1 + \text{si} x_0 x_2 \\ &\quad + z_{5,0}^2 \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \text{co} x_0^2 + \dots \end{aligned}$$

These series are also elements of $\mathbb{R}_{GC}^2 \langle \langle X \rangle \rangle$ since again all sines and cosines in the numerator can be bounded by 1 so that conservative growth constants are $K_c = \max\{z_{1,0}, z_{2,0}, z_{5,0}\}$ and $M_c = 2.4z_{5,0} \sec(0.7z_{4,0})$. The decoupling matrix

$$A = (C, \emptyset) = \begin{pmatrix} z_{5,0} \text{si} & \text{co} \\ z_{5,0} \text{co} & \text{si} \end{pmatrix} \quad (7)$$

is clearly nonsingular as long as $z_{5,0} \neq 0$.

Given a desired output function

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!}, \quad (8)$$

where $c_y = [c_{y_1}, c_{y_2}]^T$ is the generating series of y , the left inverse $c_u = [c_{\bar{u}_1}, c_{u_2}]^T$ is computed directly from (3)-(4). It is sufficient here to consider polynomial outputs up to degree three, so let $(c_{y_j}, x_0^i) = v_{ij}$ for $i = 0, 1, 2, 3$ and $j = 1, 2$. The series

$$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} := C^{\sqcup -1} \sqcup (x_0^r)^{-1} (c - c_y)$$

is found to be

$$\begin{aligned} d_1 &= \frac{v_{12} \text{si}}{z_{5,0}} - \frac{v_{22} \text{co}}{z_{5,0}} + z_{5,0} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \\ &\quad + \left(\left(\frac{v_{13}}{z_{5,0}} + v_{22} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \right) \text{si} \right. \\ &\quad \left. + \left(v_{12} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} - \frac{v_{23}}{z_{5,0}} \right) \text{co} \right) x_0 \\ &\quad + \left(\frac{v_{22} \text{si}}{z_{5,0}} + 1.7z_{5,0} \frac{\cos(1.7z_{4,0})}{\cos(0.7z_{4,0})} \right. \\ &\quad \left. + 0.7z_{5,0} \frac{\sin(1.7z_{4,0}) \tan(0.7z_{4,0})}{\cos(0.7z_{4,0})} + \frac{v_{22} \text{co}}{z_{5,0}} \right) x_1 \\ &\quad + \left(\frac{v_{22} \text{co}}{z_{5,0}^2} - \frac{v_{12} \text{co}}{z_{5,0}^2} + \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \right) x_2 + \dots, \\ d_2 &= -v_{12} \text{co} - v_{22} \text{si} \\ &\quad - \left(v_{13} \text{co} - z_{5,0} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} (v_{22} \text{co} + v_{12} \text{si}) + v_{23} \text{si} \right) x_0 \\ &\quad + (v_{12} \text{si} - v_{22} \text{co}) x_1 + \dots \end{aligned}$$

The composition inverse of d is computed componentwise using the recursive method in [2], [6], [8]. In which case, the formulas for $c_{\bar{u}_1}$ and c_{u_2} are, respectively,

$$\begin{aligned} c_{\bar{u}_1} &= (d_1^{o-1})_N = \frac{v_{22} \text{si}}{z_{5,0}} - \frac{v_{12} \text{co}}{z_{5,0}} - z_{5,0} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \\ &\quad + \frac{1}{z_{5,0}^2} \left(-2v_{12}v_{22} \cos(2(z_{3,0} + z_{4,0})) \right. \\ &\quad \left. + 1.7z_{5,0}^4 \frac{\sin(1.7z_{4,0}) \cos(1.7z_{4,0})}{\cos(0.7z_{4,0})^2} - v_{13}z_{5,0} \text{si} \right. \\ &\quad \left. + 1.7v_{12}z_{5,0}^2 \frac{\cos(1.7z_{4,0})}{\cos(0.7z_{4,0})} \text{si} - v_{22}z_{5,0}^2 \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} \text{si} \right. \\ &\quad \left. + v_{12}^2 \sin(2(z_{3,0} + z_{4,0})) - v_{22}^2 \sin(2(z_{3,0} + z_{4,0})) \right. \\ &\quad \left. + 0.7z_{5,0}^4 \frac{\sin(1.7z_{4,0})^2 \tan(0.7z_{4,0})}{\cos(0.7z_{4,0})^2} \right) \end{aligned}$$

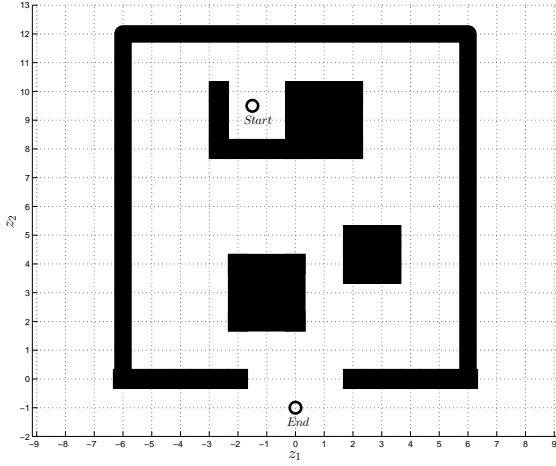


Fig. 3. Map of area with obstacles

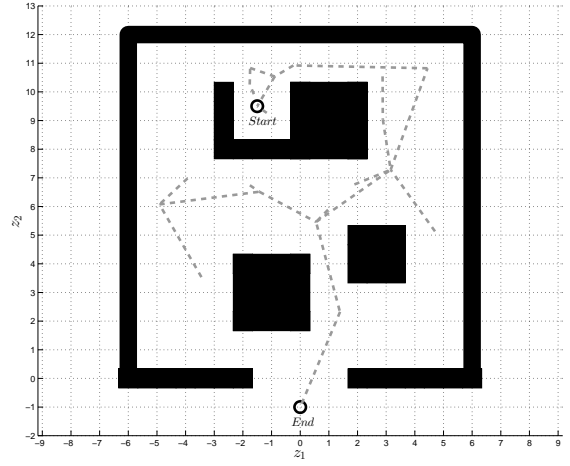


Fig. 4. RRT used to identify the shortest path

$$\begin{aligned}
& + z_{0,5} \cos \left(v_{23} - 1.7v_{22}z_{5,0} \frac{\cos(1.7z_{4,0})}{\cos(0.7z_{4,0})} \right. \\
& \left. - z_{5,0} \frac{\sin(1.7z_{4,0})}{\cos(0.7z_{4,0})} (v_{12} + 0.7v_{22} \tan(0.7z_{4,0})) \right) x_0 + \dots \\
c_{u_2} &= (d_2^{\circ-1})_N = v_{22}\sin + v_{12}\cos + \frac{1}{z_{5,0}} \left(\frac{v_{12}^2}{2} + \frac{v_{22}^2}{2} \right. \\
& + v_{13}z_{5,0}\cos + \frac{(v_{22}^2 - v_{12}^2)}{2} \cos(2(z_{3,0} + z_{4,0})) + v_{23}z_{5,0}\sin \\
& \left. - v_{12}v_{22}\sin(2(z_{3,0} + z_{4,0})) \right) x_0 + \dots
\end{aligned}$$

Some key features of these expressions are:

- i. The formulas are exact if not truncated. But, of course, truncation is necessary for implementation. The truncation gives the degree of approximation. Here the degree of approximation for $c_{\bar{u}_1} = (d_1^{\circ-1})_N$ and $c_{u_2} = (d_2^{\circ-1})_N$ means their truncation is to degree 6. These truncated inputs are then fed into the 8th degree approximation of series c via the composition product described in [8].
- ii. The formulas only need to be computed once. They can therefore be loaded directly into a micro-controller inside the vehicle. Then based on measurements of the current position and steering angle, the input for tracking the next section of the desired path can be quickly computed by just numerically evaluating the formula.
- iii. Flat outputs are not required for computing these left inverses.
- iv. One can increase the degree of approximation of the output tracking by including more input terms if the computing power is available.

IV. COLLISION AVOIDANCE SYSTEM

A collision avoidance system is now described based on the left inversion input formula for the bi-steerable car model developed in the previous section. The basic steps of the algorithm are as follows:

- i. Map the area and obstacles where the bi-steerable car moves and provide a *start* location and an *end* location.

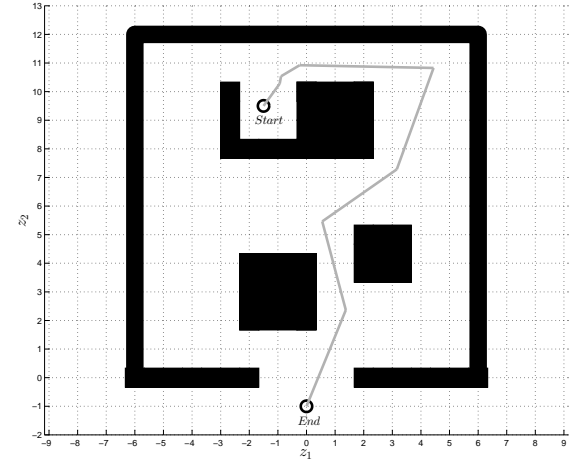


Fig. 5. Shorted path extracted from the RRT

- ii. The RRT path planning algorithm described in [16] then uses the data from the previous step to compute the path between the start and end locations by first computing a tree and then identifying the shortest path within the tree from the start location to the end location.
- iii. The shortest path is then smoothed and approximated by a cubic spline.
- iv. Each spline section, which is a cubic polynomial, is fed to the left inversion formula in order to compute the corresponding control inputs \bar{u}_1 and u_2 .
- v. The inputs \bar{u}_1 and u_2 are finally used to drive the bi-steerable car.

Consider the following specific example:

- i: Suppose the area with obstacles is the one shown in Figure 3. Pick the start and end locations to be:

$$\begin{aligned}
Start &: (-1.5, 9.5) \\
End &: (0, -1).
\end{aligned}$$

- ii: The RRT generated for this map is shown in Figure 4. The shortest path between the given start and end locations exacted from this RRT is shown in Figure 5.

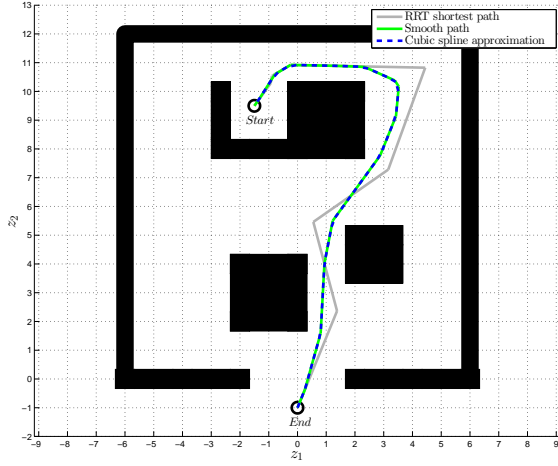


Fig. 6. Shortest, smoothed and spline approximated paths

iii: The shortest path is then smoothed and approximated by cubic splines as shown in Figure 6. The constant terms of each cubic spline section must coincide with the final position of the previous section. This means that one does not need to fit the constant terms of the spline approximation. That is, the condition $(c, \emptyset) = (c_y, \emptyset)$ in Theorem 1 is always satisfied by the cubic splines in the path approximations. The linear coefficient of the spline approximation is also subject to the range conditions of Theorem 1. However, since z_4 and z_5 in (6) are just the integrals of u_2 and \bar{u}_1 , respectively, their corresponding initial conditions $z_{4,0}$ and $z_{5,0}$ can be chosen arbitrarily as long as $z_{5,0} \neq 0$, otherwise the decoupling matrix (7) is singular. Observe that

$$(c, x_0) = \begin{pmatrix} z_{5,0} \cos(z_{3,0} + z_{4,0}) \\ z_{5,0} \sin(z_{3,0} + z_{4,0}) \end{pmatrix}, \quad (9)$$

where $z_{3,0}$ is known and fixed. Hence, one can always find $z_{4,0}$ and $z_{5,0}$ such that (c, x_0) matches the desired coefficient (c_y, x_0) , which comes from the spline approximation. Here the time interval for the simulation have been normalized to $[0, 1]$ since only the kinematics are being considered.

iv: The control inputs are now computed section by section. The first spline section has initial conditions

$$z_{1,0} = -1.5, \quad z_{2,0} = 9.5, \quad z_{3,0} = 1.5,$$

and the desired outputs of the path are

$$c_{y_1} = -1.5 + 10.75x_0 - 28.15x_0^2 + 185.15x_0^3$$

and

$$c_{y_2} = 9.5 + 14.40x_0 + 13.07x_0^2 - 24.89x_0^3.$$

As expected, the constant coefficients are automatically equal to the start position of the car. Also, (9) is solved for $z_{4,0}$ and $z_{5,0} \neq 0$ so that $(c_1, x_0) = (c_{y_1}, x_0) = 10.75$ and $(c_2, x_0) = (c_{y_2}, x_0) = 14.40$, which gives

$$z_{4,0} = -4.00028, \quad z_{5,0} = -17.97.$$

This solution is not unique. Since the overall time of the simulation was normalized, the time interval for each section

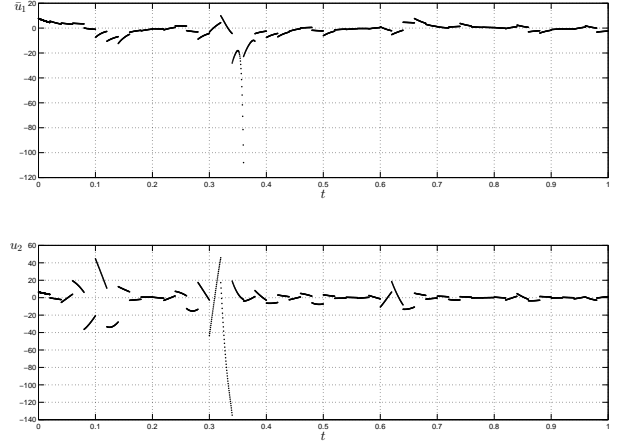


Fig. 7. Control inputs \bar{u}_1 and u_2

of the path was chosen to have duration $t_{sim} = 0.02$. The left inverse formulas in this case give

$$\begin{aligned} \bar{u}_1(t) = & 7.74 - 195.52t + 1592.16t^2 + 12925.9t^3 \\ & - 443203.0t^4 + 4.26 \times 10^6 t^5 + 2.83 \times 10^6 t^6 \end{aligned}$$

and

$$\begin{aligned} u_2(t) = & 6.36 - 142.17t + 386.62t^2 - 226.02t^3 \\ & - 2648.52t^4 + 1280.47t^5 + 76475.8t^6. \end{aligned}$$

Since the degree of the approximation was chosen to be 6, these inputs give the following errors:

$$\begin{aligned} y_1(t) - \hat{y}_1(t) = & 1.33 \times 10^6 t^7 - 5.10 \times 10^6 t^8 \\ & - 2.02 \times 10^7 t^9 + 3.47 \times 10^7 t^{10} + \dots \end{aligned}$$

and

$$\begin{aligned} y_2(t) - \hat{y}_2(t) = & -1.24 \times 10^6 t^7 + 4.39 \times 10^6 t^8 \\ & + 2.09 \times 10^7 t^9 - 2.82 \times 10^7 t^{10} + \dots, \end{aligned}$$

where y_i is the i -th component of y having generating series c_y , and $\hat{y}_i = c \circ c_u$ is truncated to degree 6. The computed control inputs are shown in Figure 7.

iv: Finally, the computed control inputs \bar{u}_1 and u_2 are used to drive the bi-steerable car as shown in Figure 8. In the same figure, the bi-steerable car path is overlapped, as comparison, with the cubic spline approximation of the shortest path (avoiding obstacles) computed by the RRT algorithm. The degree of approximation and smoothness of the bi-steerable car path can be tuned by increasing or decreasing the number of partitions of the cubic spline approximation, and the degree of approximation of the computed left inverses. Tracking performance will ultimately be bounded by the amount of computational power available.

V. CONCLUSIONS

A collision avoidance system was described for a bi-steerable car based on a left inversion formula for Fliess operators whose generating series have a well defined vector relative degree. This allows one to integrate directly the

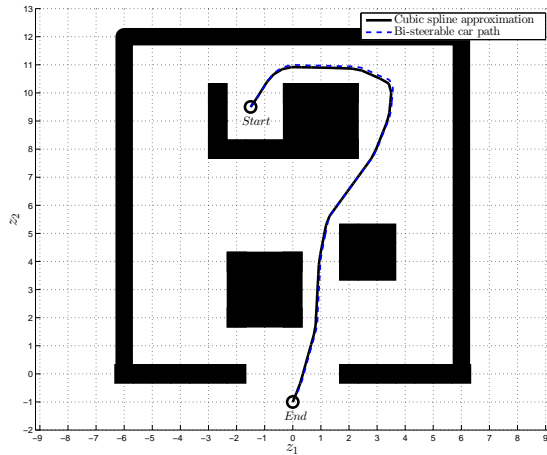


Fig. 8. Cubic spline approximated path and bi-steerable car trajectory

kinematics of the car and a path planning algorithm without the need for passing through a flat output as is done in other methods. In principle, the full dynamics could also be inverted to give a more realistic collision avoidance system. The method was demonstrated by numerical simulation.

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